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THE CARDINALITY OF MANIFOLD ATLASES

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ABSTRACT

It is shown that any finite dimensional C^0 manifold (connected and Hausdorff but otherwise unrestricted) has an atlas of cardinality not greater than that of the continuum; while if it has a Hölder continuous pseudo-Riemannian metric then there is a countable atlas.

1. Introduction and results

By a manifold (respectively, C^1 manifold) we shall mean a pair $\mathcal{M} = (M, \mathcal{A})$, where M is a connected, Hausdorff topological space and the atlas $\mathcal{A} = (U_\iota, p_\iota)_{\iota \in I}$ comprises an open cover $(U_\iota)_{\iota \in I}$ of unrestricted cardinality and injective coordinate maps $p_\iota : U_\iota \to \mathbf{R}^n$ (n fixed, finite) such that $\phi_{ij} := p_i p_j^{-1}$, where defined, is continuous (respectively C^1 with differential $D\phi_{ij}$). The equivalence of atlases is defined as usual, and we write

$$|\mathcal{M}| := \min\{k : \mathcal{M} \text{ has an equivalent atlas of cardinality } k\}.$$

It is well-known that there exists a C^1 $\mathcal{M}$ with $|\mathcal{M}| = c$ (where $c = 2^d = \operatorname{card}(\mathbf{R})$; $d = \operatorname{card}(\mathbf{Z})$) and that if $\mathcal{M}$ has a C^0 Riemannian structure or a C^1 pseudo-Riemannian structure then $|\mathcal{M}| \leqq d$. Geroch [1] published the pseudo-Riemannian result for C^3 metrics; the simplest proof for C^1 metrics follows on noting (cf. Schmidt [2]) that a C^1 pseudo-Riemannian structure induces a C^0-Riemannian structure on the frame bundle.

We shall formulate a more general condition under which $|\mathcal{M}| \leqq d$ by defining a *para-linear structure* on $\mathcal{M} = (M, \mathcal{A})$ to be a family $(\mathcal{A}_\varepsilon)_{\varepsilon \in \mathbf{R}}$ of C^1 atlases equivalent to $\mathcal{A}$ such that if (U_i, p_i) and (U_j, p_j) belong to $\mathcal{A}_\varepsilon$, then there exists a linear map $L \in \mathrm{GL}(n, \mathbf{R})$ with

$$\|L \circ D\phi_{ij}(x) - 1\| < \varepsilon \qquad \text{for all } x \in p_j(U_i \cap U_j)$$

(where $\|\ \ \|$ denotes the mapping norm). We shall prove the following.

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THEOREM. (i) *For any manifold* $\mathcal{M}$, $|\mathcal{M}| \leqq c$.
(ii) *If* $\mathcal{M}$ *has a para-linear structure then* $|\mathcal{M}| \leqq d$.

COROLLARY. *If* $\mathcal{M}$ *has a* $C^{1,\alpha}$ *atlas and a* $C^{0,\alpha}$ *pseudo-Riemannian structur then* $|\mathcal{M}| \leqq d$, *and so* M *is paracompact.*

Here $C^{k,\alpha}$ *denotes functions with Hölder continuous k'th derivatives.*

The main point of the paper is the reduced differentiability used in th corollary, which has useful applications to models in General Relativity contair ing impulse-waves. Part (i) of the theorem is included for completeness: thoug simple to prove and accepted folklore, the author has not found the result in th literature. The proof of (ii) owes much to Geroch [1].

2. Proof of Theorem (i)

We define by induction an increasing collection of submanifolds $\mathcal{M}_\alpha$ $(M_\alpha, \mathcal{A}_\alpha)$ for ordinals α, showing that $\mathrm{card}(\mathcal{A}_\alpha) \leqq c$ for $\alpha \leqq \Omega$ and that $M_\Omega = l$ (where Ω is the first uncountable ordinal).

Choose $\mathcal{M}_0 = (U_0, (U_0, p_0))$ for any $(U_0, p_0) \in \mathcal{A}$. If κ is a limit ordinal, the define $\mathcal{M}_\kappa = (\bigcup_{\alpha<\kappa} M_\alpha, \bigcup_{\alpha<\kappa} \mathcal{A}_\alpha)$. The inductive hypothesis "$\mathrm{card}(\mathcal{A}_\alpha) \leqq c$ fc $\alpha < \kappa$" then gives that $\mathrm{card}(\mathcal{A}_\kappa) \leqq c$ provided $\kappa \leqq \Omega$.

For a successor ordinal $\beta + 1$, begin by defining $\mathcal{A}^*_\beta$ to be the complete atlɛ equivalent to $\mathcal{A}_\beta$. That is, $\mathcal{A}^*_\beta = \mathcal{A}_\beta \cup \mathcal{A}^\dagger_\beta$ where $\mathcal{A}^\dagger_\beta$ is the set of all charts $(U, p$ with $U \subset M_\beta$ and pp_j^{-1} being a local homeomorphism (where defined) for a $p_j \in \mathcal{A}_\beta$. The elements of $\mathcal{A}^\dagger_\beta$ are completely characterised by the functions pp (regarded as $\varnothing$ if the domain is empty) as j ranges over an index set for $\mathcal{A}_\beta$; bı $\mathrm{card}(\mathcal{A}_\beta) \leqq c$ for $\beta < \Omega$ (inductive hypothesis) and the set of local homeomorphic functions from open sets of $\mathbf{R}^n$ into $\mathbf{R}^n$ has cardinality c; henc the functions pp_j^{-1} form a set of cardinality c and $\mathrm{card}(\mathcal{A}^*_\beta) = c$.

Let $\mathcal{A}^*$ be the complete atlas equivalent to $\mathcal{A}$. For each $(U, p) \in \mathcal{A}^*_\beta$, s $j(U, p)$ to be any pair $(U', p') \in \mathcal{A}^*$ with $\bar{U} \subset U'$ and $U' \not\subset M_\beta$, with $p' \mid U = p$, such a pair exists; otherwise define $j(U, p) = (U, p)$. Finally, ɛ $\mathcal{A}_{\beta+1} := \{j(U, p) : (U, p) \in \mathcal{A}^*_\beta\}$ and $M_{\beta+1} := \bigcup_{(U,p) \in \mathcal{A}^*_\beta} U'$, $\mathcal{M}_{\beta+1} := (M_{\beta+1}, \mathcal{A}_{\beta+}$ This gives $\mathrm{card}(\mathcal{A}_{\beta+1}) = \mathrm{card}(\mathcal{A}^*_\beta) = c$, by induction, for $\beta < \Omega$.

To show that $M_\Omega = M$, suppose the contrary. Then, since M is connected ar M_Ω is open, there is an $x \in \bar{M}_\Omega \backslash M_\Omega$ and some $(U, p) \in \mathcal{A}$ with $x \in U$. The s $p(U \cap M_\Omega) \subset \mathbf{R}^n$ can be covered by an increasing countable collection $(V_i)_{i \in \omega}$ · open sets with compact closures $\bar{V}_i \subset V_{i+1}$. But $M_\Omega = \bigcup_{\alpha<\Omega} M_\alpha$ and so ł compactness there exist ordinals $\alpha_i < \Omega$ $(i \in \omega)$ such that each $V_i \subset p(U \cap M_{\alpha_i}$ Let $\beta = \bigcup_{i \in \omega} \alpha_i$. Then $\mathrm{card}(\beta) = d$ and so $\beta < \Omega$. Since $V_i \subset p(U \cap M_\beta)$ for ː

i, $U \cap M_\beta = U \cap M_\Omega$. But now $p \mid U \cap M_\beta$ is a coordinate map on M_β and thus the pair $W = (U \cap M, p \mid U \cap M_\beta)$ is in $\mathscr{A}^*_\beta$. So the chart $j(W) = (U', p')$ has $x \in \overline{U \cap M_\beta} \subset U'$ (by construction of j) and so $x \in M_{\beta+1} \subset M_\Omega$, a contradiction.

3. Proof of Theorem (ii)

Let $\mathscr{A}^*_\varepsilon$ be the atlas obtained by adjoining to $\mathscr{A}_\varepsilon$ all charts of the form (U', p'), where $p' = L \circ (p \mid U') + a$ for any $(U, p) \in \mathscr{A}_\varepsilon$ and $U' \subset U$, $L \in \mathrm{GL}(n, \mathbf{R})$, $a \in \mathbf{R}^n$. Clearly $(\mathscr{A}^*_\varepsilon)_{\varepsilon \in \mathbf{R}}$ is still a para-linear structure. Let $\tilde{f}_\xi$ be the frame $(\partial / \partial x^r)^n_{r=1}$ at $\xi \in \mathbf{R}^n$, and suppose f is some frame on $\mathscr{M}$. Then by "an $\mathscr{A}^*_\varepsilon$ r-ball about f" we mean a triple (U, f, p), where $(U, p) \in \mathscr{A}^*_\varepsilon$, $p_* f = \tilde{f}_0$ and $p(U) = \{\xi \in \mathbf{R}^n : |\xi| < r\}$. Choose countable dense sets $(z_\iota)_{\iota \in \omega}$ and $(l_\iota)_{\iota \in \omega}$ in the unit n-ball $(\{\xi \in \mathbf{R}^n : |\xi| < 1\})$ and in $\mathrm{GL}(n, \mathbf{R})$, respectively. Let $l_i f$ for any frame $f = (\underset{1}{f}, \cdots, \underset{n}{f})$ be the frame $((l_i)_1{}^s \underset{s}{f}, \cdots, (l_i)_n{}^s \underset{s}{f})$. If $\mathscr{B} = (U, f, p)$ is an r-ball, define $X_{\mathscr{B}}$ as the set $\{p_x^{-1}(l_i \tilde{f}_{z_j}) : i, j \in \omega\}$.

As in the proof of (i) we define a sequence of submanifolds $\mathscr{M}_\alpha = (M_\alpha, \mathscr{A}_\alpha)$ for ordinals $\alpha \leqq \omega$, where $\mathscr{A}_\alpha = \{(U, p) \in \mathscr{A}^*_\varepsilon : U \subset M_\alpha\}$, and ε is fixed at a sufficiently small value ($\varepsilon = 1/7$ will do). Each $\mathscr{M}_\alpha$ will be furnished with a countable dense set X_α of frames (i.e. X_α is dense in the frame bundle of $\mathscr{M}_\alpha$).

To start, M_0 is taken to be any U for which there exists an $\mathscr{A}^*_\varepsilon$ r-ball $\mathscr{B} = (U, f, p)$; X_0 is defined as $X_{\mathscr{B}}$.

For a limit ordinal κ, $M_\kappa = \bigcup_{\gamma < \kappa} M_\gamma$ and $X_\kappa = \bigcup_{\gamma < \kappa} X_\gamma$ (though only the case $\kappa = \omega$ will concern us).

For a successor ordinal $\beta + 1$, proceed as follows. For any frame f on $\mathscr{M}$ define $\rho(f) = \min(1, \tfrac{1}{2}\sup\{r : \exists \text{ an } \mathscr{A}^*_\varepsilon\ r\text{-ball about } f\})$. Now, for each $f \in X_\beta$, choose an $\mathscr{A}^*_\varepsilon$ $\rho(f)$-ball $\mathscr{B}(f) = (U_f, f, p_f)$ and set

$$M_{\beta+1} := \bigcup_{f \in X_\beta} U_f, \quad X_{\beta+1} := \bigcup_{f \in X_\beta} X_{\mathscr{B}(f)}.$$

We now show that $M_\omega = M$, by a method similar to that used for (i). Suppose not, and take $x_0 \in \bar{M}_\omega \setminus M_\omega$. Choose some chart $(U, p) \in \mathscr{A}_\varepsilon$ with $x_0 \in U$ and a sequence $(f_\iota)_{\iota \in \omega}$ with $f_\iota \in X_\omega$ and $\pi f_\iota \to x_0$ (where $\pi : LM \to M$ is the projection of the frame bundle). Since each $f \in X_\omega$ belongs to a family (obtained by varying i in the definition of $X_{\mathscr{B}}$) which is dense in the fibre containing f, we can choose other members f'_ι of the families containing the f_ι so that $\pi f'_\iota = \pi f_\iota \to x_0$ and

$$\|\mathbf{f}'_\iota - \mathbf{1}\| < \varepsilon \tag{1}$$

where $\mathbf{f}'_\iota$ is the matrix of components of the members of f_ι with respect to the coordinates p.

Set $R = \min\{1, \inf\{|\xi - p(x_0)| : \xi \in \mathbf{R}^n \setminus p(U)\}\}$; and choose i_1 so large that

$$|p(x_1) - p(x_0)| < R/40 \tag{2}$$

(where $x_1 = \pi f_{i_1}$) and so that $p(x_1)$ can be joined to $p(x_0)$ by a straight line ir $p(U)$. Set f for f'_{i_1}, f for f'_{i_1}.

If we define $p'(x) = f^{-1}(p(x) - p(x_1))$ then

$$|p'(x)| < R/4 \Rightarrow |p(x) - p(x_1)| \leqq \|f\| R/4 \leqq R/2$$

(provided $\varepsilon < 1$)

$$\Rightarrow |p(x) - p(x_0)| \leqq R \qquad \text{(from (2))}.$$

It follows that U contains the $\mathcal{A}^*_\varepsilon$ $R/4$-ball (B', f, p'), where $B' = p'^{-1}\{x : |x| < R/4\}$, so that, by definition of ρ,

$$\rho(f) > R/8. \tag{3}$$

Certainly $x_1 \in M_\alpha$ for some $\alpha < \omega$; so consider the $\rho(f)$-ball (B'', f, p'') used ir the definition of $M_{\alpha+1}$. Let $\xi : [0,1] \to \mathbf{R}^n$ be the line $\xi(\lambda) = (1-\lambda)p(x_1) + \lambda p(x_0$ from x_1 to x_0. Then $p^{-1}\xi(\lambda)$ lies in B'' for small enough λ and (since $x_0 \notin M_{\alpha+1}$ leaves B'' for the first time at some $\lambda_1 < 1$. Noting that $p''\xi(0) = 0$, we have

$$|p'p^{-1}\xi(1)| < \int_0^\lambda \left| \frac{d}{d\lambda}(p''p^{-1}\xi(\lambda)) \right| d\lambda \quad \text{for } \lambda < \lambda_1$$

$$< \int_0^\lambda \|D\phi(\xi(\lambda))\| d\lambda \,.\, |p(x_0) - p(x_1)|$$

where $\phi = p'' \circ p^{-1}$, i.e.

$$|p''p^{-1}\xi(\lambda)| < \lambda \|L^{-1}\|(1+\varepsilon)|p(x_0) - p(x_1)| < 2\|L^{-1}\| R/40 \tag{4}$$

from (2), provided $\varepsilon < 1$ and $L \in \mathrm{GL}(n, \mathbf{R})$ is such that $\|LD\phi - 1\| < \varepsilon$. Such ai L exists since p and p'' are coordinate maps of charts in $\mathcal{A}^*_\varepsilon$.

Now $p''_* f = \tilde{f}_0$ and $p_* f = f\tilde{f}_0$. Hence at the point $p(x_0)$ we have $D\phi$ = $D(p'' \circ p^{-1}) = f^{-1}$. But $\|LD\phi - 1\| < \varepsilon$, by the choice of L, so that using (1) simple estimation procedure gives $\|L - 1\| < 7\varepsilon/2$ provided $\varepsilon < 1/4$ and s $\|L^{-1}\| \leqq 2$ provided $\varepsilon < 1/7$. Thus (4) gives $|p''p^{-1}\xi(\lambda)| \leqq R/10$. But, at λ $p^{-1}\xi(\lambda)$ leaves the ball B'' of coordinate (p'') radius at least $R/8$ (from (3)), i.e $|p''p^{-1}\xi(\lambda_1)| > R/8$, a contradiction.

4. Proof of Corollary

LEMMA. *Let $B_t = \{\xi \in \mathbf{R}^n : |\xi| < t\}$ and let U be a connected open set in $\mathbf{R}^n$ containing $\bar{B}_\tau$ for some τ, $0 < \tau < 1$. Let $x' : U \to \mathbf{R}^n$ be a $C^{1,\alpha}$ diffeomorphism $(0 < \alpha < 1)$ with $Dx'(0) = 1$. Let there be given a $C^{0,\alpha}$ pseudo-Riemannian metric on U with a matrix of components g with respect to natural coordinates and g' with respect to coordinates induced by x' (viz. $x'^*(g'_{ij}(x'(\xi))dx^i \otimes dx^j) = g_{ij}(\xi)dx^i \otimes dx^j$). Let β be any real number $0 < \beta < \alpha$. Suppose $\|g(x_1) - g(x_2)\| < \bar{\varepsilon}|x_1 - x_2|^\beta$ and $\|g'(x'_1) - g'(x'_2)\| < \bar{\varepsilon}|x'_1 - x'_2|^\beta$ for all $x_1, x_2 \in B_\tau$ and $x'_1, x'_2 \in x'(B_\tau)$. Then for all $\varepsilon > 0$ there is a number ε_0, depending only on α, β, ε and n, such that, if $\bar{\varepsilon} < \varepsilon_0$, then $\|Dx' - 1\| < \varepsilon$ in $B_{\tau/2}$.*

PROOF OF LEMMA. Write L for $(Dx')^{-1}$. Let $L_A = L(x_A)$, $g_A = g(x_A)$, $g'_A = g'(x'_A)$ $(A = 1, 2)$, where $x_1, x_2 \in B_{\tau'}$ $(\tau' < \tau)$, and $x'_A = x'(x_A)$. Then direct calculation gives

$$L_2^T(g_1 - g_2)L_2 + (g'_2 - g'_1) = \Lambda^T g'_1 + g'_1\Lambda + \Lambda^T g'_1 \Lambda \tag{5}$$

with $\Lambda = L_1^{-1}(L_2 - L_1)$ and ^{T} denoting matrix transpose. For small enough τ' (a restriction to be removed at the end of the proof), since $L \in C^{0,\alpha}$ and $L(0) = 1$ we can ensure that $\|L_A - 1\| < \frac{1}{2}$, $\|\Lambda\| < \sqrt{\bar{\varepsilon}}\,\bar{\delta}^{\beta/2}$ (where $\bar{\delta} = |x_1 - x_2|$). In that case $|x'_1 - x'_2| < 2\bar{\delta}$, and taking norms in (5) gives

$$|\Lambda^T g'_1 + g'_1\Lambda| \leqq 8\bar{\varepsilon}\delta^\beta \tag{6}$$

(where $|A|$ for a matrix A denotes $\max|A^i_j|$).

Now choose a C^∞ function $\chi : \mathbf{R}^n \to \mathbf{R}$ with support in B_1 such that $\int \chi d^n x = 1$ and set

$$\psi^i_j(y) = \int_U \delta^{-n}\chi\left(\frac{x - y}{\delta}\right) L^i_j(x)d^n x$$

(where $\delta > 0$ is to be determined shortly). Then

$$C_{ijk} := L^m_i g_{mn}\psi^n_{j,k} = -\delta^{-(n+1)}\int_U \chi_{,k}(g'(x'(\xi))\Lambda)_{ij}d^n\xi$$

from which (6) gives

$$|C_{(ij)k}| \leqq 4\kappa\bar{\varepsilon}\delta^{\beta-1} \tag{7}$$

where

$$\kappa = \max_i \int |\chi_{,i}(\xi)|d^n\xi \quad \text{and} \quad C_{(ij)k} = \tfrac{1}{2}(C_{ijk} + C_{jik}).$$

The definition of ψ can be rewritten (integrating by parts) to yield

$$\psi^{i}_{j,k} = \delta^{-(n+1)} \int_U \chi_{,kj} x^i d^n x$$

whence symmetry on j and k gives us $C_{ijk} = C_{ikj}$. Consequently

$$C_{ijk} = C_{(ik)j} - C_{(kj)i} + C_{(ji)k}$$

and so from (7)

$$|C_{ijk}| \leqq 12k\bar{\varepsilon}\delta^{\beta-1}$$

and hence, from the definition of C,

$$|\psi^{i}_{j,k}| \leqq 48\kappa\bar{\varepsilon}\delta^{\beta-1}. \tag{8}$$

The Hölder condition on L gives us, within $B_{\tau'/2}$, that

$$|L(y) - \psi(y)| \leqq K_{(\alpha)}\delta^{\alpha} \tag{9}$$

where

$$K(\alpha) \geqq \sup\{|L_1 - L_2|\,|x_1 - x_2|^{-\alpha} : x_1, x_2 \in B_{\tau'}, x_1 \neq x_2\} \tag{10}$$

provided $\delta < \tau'/2$; thus, combining (9) and (8) gives

$$|L_1 - L_2| \leqq 2K(\alpha)\delta^{\alpha} + K's\delta^{\beta-1} \tag{11}$$

where

$$K' = 48\kappa\bar{\varepsilon}, \qquad s = |x_1 - x_2|.$$

We now choose δ so as to minimise the right-hand side of (11), viz.

$$\delta = \left[\frac{(1-\beta)K's}{2\alpha K(\alpha)}\right]^{1/(\alpha-\beta+1)}$$

which satisfies $\delta < \tau'/2$ for small enough $\bar{\varepsilon}$ (and so small enough K'), given tha $s < \tau' < 1$ and $K(\alpha)$ can be taken greater than unity. Substituting in (11) gives

$$|L_1 - L_2| \leqq (K's)^{\rho(\alpha)} K(\alpha)^{\sigma(\alpha)} J_{\alpha,\beta} \tag{12}$$

where

$$\rho(\alpha) = \alpha/(\alpha-\beta+1), \qquad \sigma(\alpha) = (1-\beta)/(\alpha-\beta+1) \qquad \text{and}$$

$$J_{\alpha,\beta} = \left[2\left(\frac{1-\beta}{2\alpha}\right)^{\rho(\alpha)} + \left(\frac{2\alpha}{1-\beta}\right)^{\sigma(\alpha)}\right].$$

Equation (12) expresses a Hölder condition on L with exponent $\rho(\alpha)$. This enables us to base an iterative procedure on equations (12) and (10): the condition (12) allows us to compute a value for $K(\rho(\alpha))$ by using equality in (10); we then replace α by $\rho(\alpha)$ in (12) to give a Hölder condition with exponent $\rho(\rho(\alpha))$, and so on. The successive Hölder exponents are thus given by the recurrence relation

$$\alpha_n = \rho(\alpha_{n-1}) \qquad (n = 1, 2, \cdots),$$
$$\alpha_0 = \alpha$$

with solution $\alpha_n = \alpha\beta/[(\beta - \alpha)(1-\beta)^n + \alpha]$. This function decreases monotonically to β as $n \to \infty$, although the iterative process is only admissible so long as $\delta < \tau'/2$ is maintained.

The relations (12) and (10) give, by this method,

$$K(\alpha_n) = \eta_{\alpha_n}(K(\alpha_{n-1}))^{\sigma(\alpha_{n-1})}, \qquad \eta_\alpha = J(\alpha, \beta)K'^{\rho(\alpha)}.$$

Since, as may easily be verified, $\Pi\,\sigma(\alpha_n)$ diverges to zero while $J(\alpha, \beta)$ tends to $J(\beta, \beta)$, this implies that $K(\alpha_n) \to 0$ provided $\bar{\varepsilon}$ is small enough to make $\eta_\alpha < 1$ for all α_n.

We perform the iteration, reducing $K(\alpha_n)$ until the condition $\delta < \tau'/2$ is violated, at which point

$$K(\alpha_n) \leqq \frac{(1-\beta)K'}{4\alpha(\tau'/2)^{\alpha_n - \beta}} .$$

We can now choose $\bar{\varepsilon}$ small enough so that $K(\alpha_n)\tau'^{\alpha_n}$ will be less than 1/2 for all $\tau' < 1$, and so that $K(\alpha_n)\tau'^{\alpha_n - \beta}$ is less than $\sqrt{\bar{\varepsilon}}$. This means that the procedure will be valid for all $\tau' < \tau < 1$, as required.

Thus we derive a Hölder condition with exponent β on L in B_τ with constant depending only on α, and which can be made arbitrarily small by choice of $\bar{\varepsilon}$. The result then follows.

Proof of Corollary, continued. The construction of an atlas providing a para-linear structure for $\mathcal{M}$ is now immediate: the Hölder condition on the metric ensures that, for any point $x \in M$, we can choose a neighbourhood U of x and a coordinate map such that the coordinate-image satisfies the conditions of the Lemma. Restricting U down to $p^{-1}(B\tau'/2)$ then gives a chart, the collection of all such charts providing (according to the Lemma) the required para-linear structure. Part (ii) of the theorem now establishes the corollary.

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